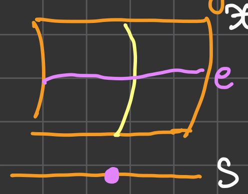
Definition: $\begin{matrix}\mathfrak{X}\\ \downarrow\\ S\end{matrix}$ is an abelian scheme over S if it is a $\overset{\text{(projective morphism)}}{\overbrace{\text{projective}}}$ (connected) group scheme over S. Group scheme: Group object in the category Sch

What does it mean?: identity element $$\begin{tikzcd} \mathfrak{X} \arrow[d, "f"'] \\ S \arrow[u, bend right, "e"'] \end{tikzcd}$$ $f\circ e = id_S$

Geometrically:



multiplication: $m: \mathfrak{X}\times_S\mathfrak{X}\to\mathfrak{X}$

associativity:

$$\begin{tikzcd} \mathfrak{X}\times_S\mathfrak{X}\times_S\mathfrak{X} \arrow[r, "id\times m"] \arrow[d, "m\times id"'] & \mathfrak{X}\times_S\mathfrak{X} \arrow[d, "m"] \\ \mathfrak{X}\times_S\mathfrak{X} \arrow[r, "m"] & \mathfrak{X} \end{tikzcd}$$

identity property: $$\begin{tikzcd} S\times_S\mathfrak{X} \arrow[r, "e\times id"] \arrow[rr, bend right, "id"'] & \mathfrak{X}\times_S\mathfrak{X} \arrow[r, "m"] & \mathfrak{X} \end{tikzcd}$$

inverse morph.: $\imath: \mathfrak{X}\to\mathfrak{X}$

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "id\times\imath"] \arrow[dr, "f"'] & \mathfrak{X}\times_S\mathfrak{X} \arrow[r, "m"] & \mathfrak{X} \\ & S \arrow[ur, "e"'] & \end{tikzcd}$$

Example:

- For $S=\mathbb{C}$: $GL(n,\mathbb{C})$, $SL(n,\mathbb{C})$, $PGL(n,\mathbb{C})$, $(\mathbb{A}^1,+)=\mathbb{G}_a$, $(\mathbb{A}^1\setminus\{0\},\cdot)=\mathbb{G}_m$.
  (not projective just group schemes).
- Projective: Elliptic curves group structure, does this underly a morphism? (yes everything is algebraic)

Definition: Abelian variety = abelian scheme / field or alg. closed field.

Future: Abelian variety / $\mathbb{C}$ is smooth (always a manifold).

Definition: X is a complex torus if it is a compact connected complex Lie group.
(just group object in the category of complex manifolds).

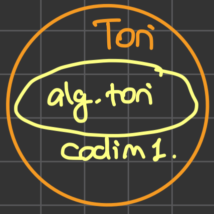
Goal 1:

1.a: classify tori 1.b: criterion when they are algebraic.

Theorem: (Chow) $X\subseteq\mathbb{P}^n_{\mathbb{C}}$ closed analytic subset defined by holomorphic functions $\Rightarrow$ it is algebraic.

1.c: understanding line bundles on tori (to understand if it embeds into $\mathbb{P}^n_{\mathbb{C}}$).

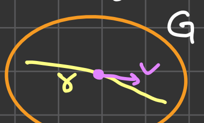
Moduli Theory:



Background on 1 parameter subgroups, exponential map:

G always a complex Lie group. [cpt, connected] (manifold + group map).

Definition: 1 parameter subgroup $\gamma:(\mathbb{C},+)\to G$ morphism of Lie gps.



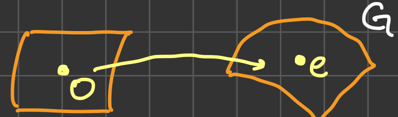
Theorem: $\forall v\in T_eG$ (e neutral element) $\exists!\,\gamma_v$ st $\gamma'(e)=v$

Example: $G=\mathbb{G}_m$ $\gamma=e^{cx}\Rightarrow\gamma'(0)=c$.

Proof: If there is such γ, $\gamma(t_0+\varepsilon)=L_{\gamma(t_0)}\circ\gamma(\varepsilon)$ with $L_{\gamma(t_0)}$ left mult. by $\gamma(t_0)\Rightarrow\gamma'(t_0)=T_eL_{t_0}(v)$, $v=\gamma'(0)$. Define $g\in G\mapsto T_eL_g(v)$ is a left invariant vector field. Let γ = integral curve of this vector field through e.

Definition: exponential map: $\overset{\mathbb{C}^n}{\overbrace{T_eG}}\to G$ morphism of complex manifolds.

$\exp(v)=\gamma_v(1)$ **Note:** $\exp(tv)=\gamma_v(t)$ **Proof:** $\exp(tv)=\gamma_{tv}(1)$, need: $\gamma_v(t)=\gamma_{tv}(1)$ more generally: $\gamma_v(ts)=\gamma_{tv}(s)$ Notice LHS and RHS are both 1 parameter subgroups in s variable. $\Rightarrow$ by unicity of Theorem above enough to show derivatives are the same $\Rightarrow$ use chain rule $\square$



Theorem: $T_0 \exp$ is invertible ⇒ exp is a local biholo.

Goal 1.a.i: • exp is the universal cover • exp. is a gpe. homom. • ker is a full rank lattice of $\mathbb{C}^n$
need compact + connected, from now on we assume these.

(after showing the goal)

Example: $G = \mathbb{C}^n/\Lambda$, for instance $n=1$, $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}$ (the elliptic curve $zy^2=(x-1)(x+1)x \subseteq \mathbb{P}^3_{\mathbb{C}}$) topologically $S^1 \times S^1$ torus. only one with 4 automorphism.

Remark: In general for $n=1$ $G=\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}.\tau$ (can biholo. rotate the lattice.)
Application: $\mathrm{Aut}(G,e)$ (as Lie groups or algebraic gps) by GAGA.
Remark: Only other one with $|\mathrm{Aut}| \neq 2$ is $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}.e^{2\pi i/6}$ which has 6 automorph.

Realizing Goal 1.a.i

Proposition: G abelian as a group
Proof: $y \mapsto xyx^{-1}$ C_x conjugation by x, morphism of holo. manifolds.
$[x \overset{Ad}{\longmapsto} C_x \in \mathrm{Aut}(G)]$ $x \xrightarrow[\phi]{} T_eC_x \in \mathrm{Aut}_{\mathbb{C}}(T_eG) \subseteq GL_n(\mathbb{C}) \subseteq \mathbb{C}^{n\times n}$
ϕ_{ij} the coord. functions : $G \to \mathbb{C}$ holomorphic map and G compact + connected
⇒ ϕ_{ij} = const. Note that $C_e = id_G$ ⇒ $T_eC_e = id_{T_eG}$ ⇒ $\phi = id_{T_eG}$ ⇒ $\forall x \in G$ $T_eC_x = id_{T_eG}$
We need $C_x = id_G$. **Fact:**

$$\begin{tikzcd} T_eG \arrow[r, "T_eC_x"] \arrow[d, "\exp"'] & T_eG \arrow[d, "\exp"] \\ G \arrow[r, "C_x"] & G \end{tikzcd}$$

commutes. (Exercise: Show it by unicity of 1-parameter subgroups).

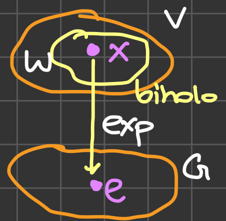
⇒ $C_x \circ \exp = \mathrm{id}$ + exp is local biholo on e ⇒ $C_x|_{\text{neigh. } U \text{ around } e} = \mathrm{id}$
(Exercise: U generates G, idea: open and closed). ⇒ $C_x = \mathrm{id}$ □

Corollary: exp is a homomorphism (not just manifold homo but also group).
Proof: $t \mapsto \exp(tx)\exp(ty) = \gamma(t)$, $t_1+t_2 \mapsto \exp(t_1x).\exp(t_2x)\exp(t_1y).\exp(t_2y) = \gamma(t_1+t_2)$ as abelian can switch middle terms. $\gamma(t)$ 1-parameter subgroup ⇒ $\gamma(t) = \exp(zt)$ with $z = \gamma'(0) = x+y$ ⇒ $\exp(t(x+y)) = \exp(tx).\exp(ty)$ plug in $t=1$. □

So far:

$$\begin{tikzcd} V=\mathbb{C}^n \arrow[d, "\exp"] \\ G \end{tikzcd}$$

Lie gpe homo. + local biholo around e] local biholo around each point



⇒ no other element in W maps to e ⇒ $\ker(\exp) = U$ is discrete.

$\mathrm{Im}(\exp) \supseteq$ a neigh. of the origin ⇒ exp surjective.

END OF LECTURE 1

Warm-up: What is U?

Lemma: $\Lambda \subseteq \mathbb{R}^n$ discrete subgroup ⇒ fin. gen.
Proof: WLOG Λ generates $\mathbb{R}^n$ (o.w replace $\mathbb{R}^n$ by $\langle\Lambda\rangle_{\mathbb{R}}$) ⇒ $v_1, -, v_n \in \Lambda$ $\mathbb{R}$-lin. basis.
for $n=2$:

by translation by v_1, v_2 elements in P generate Λ and P contains finitely many elements as Λ is discrete.

Corollary: $\Lambda \cong \mathbb{Z}^r$ as an abstract group (torsion free + fin gen).
Lemma: $r = \dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}}(\Lambda)$
Proof: for $n = \dim_{\mathbb{R}} \mathrm{Span}_{\mathbb{R}}(\Lambda) = 2$: $\Lambda \cong \mathbb{Z}^r$. Assume $r>2$ ⇒ $\exists v_3$ ⇒ v_3 not a rational lattice
$\overset{\cup\!|}{v_i} \to \overset{\cup\!|}{e_i}$ element because ow we would obtain a $\mathbb{Z}$-lin. relation between v_1, v_2, v_3. ⇒ $\exists n, m \in \mathbb{Z}$ such that $\underbrace{nv_3 - (av_1+bv_2)}_{w_1}$ and $\underbrace{mv_3 - (cv_1+dv_2)}_{w_2}$ are both in a small ball of radius $\varepsilon > 0$ for some $a,b,c,d \in \mathbb{Z}$. ⇒ $|w_1 - w_2| < \varepsilon$ can do this for all ε and $w_1 - w_2 \in \Lambda$ ⇒ contradicts discreteness of Λ.

Corollary: $\ker(\exp) \cong \mathbb{Z}^{2n}$ as an abstract group. Generator set of $\ker(\exp)$ is unique up to the action of $GL(2n,\mathbb{Z})$. Also we can assume that $v_i = e_i$ for $i=1,-,n$ where $\{v_i\}$ is a generator set.

Moduli Space (don't quote Zsolt on this): $GL(n,\mathbb{C})/GL(2n,\mathbb{Z}) \ni [v_{n+1},-,v_{2n}]$.
(Exercise: What is the action of $GL(2n,\mathbb{Z})$).

Remark: Lattice is full rank because: $X \cong_{\text{diffeo}} \mathbb{R}^{2n}/\Lambda \cong \mathbb{Z}^r$ with $r \leq 2n$. We can choose basis of $\mathbb{R}^{2n}$ such that $\underbrace{v_1,-,v_r}_{\text{basis of } \Lambda}, \underbrace{v_{r+1},-,v_{2n}}_{\text{random stuff}}$ $X \cong_{\text{diffeo}} \mathbb{R}^{2n}/\sum_{i=1}^{r}\mathbb{Z}e_i \cong (S^1)^r \times \mathbb{R}^{2n-r}$ but X is compact $\Rightarrow r = 2n$ □

Theorem: (Appel-Humbert):

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(U, C_1^*) \arrow[r] \arrow[d, leftrightarrow] & \{(H,\alpha) \mid \ldots\} \arrow[r] \arrow[d, leftrightarrow] & \left\{\begin{matrix}\text{group } H: V\times V \to \mathbb{C} \\ \text{hermitian forms st} \\ \operatorname{Im} H(U\times U) \subseteq \mathbb{Z}\end{matrix}\right\} \arrow[r] \arrow[d, leftrightarrow] & 0 \\
0 \arrow[r] & \operatorname{Pic}^0 X \arrow[r] & \operatorname{Pic} X \arrow[r, "c_1"'] & \ker(\underbrace{H^2(X,\mathbb{Z})}_{\cong H^2_{\text{sing}}(X,\mathbb{Z})} \to H^2(X,\overbrace{\mathcal{O}_X}^{\text{holo structure sheaf}})) \arrow[r] & 0
\end{tikzcd}$$

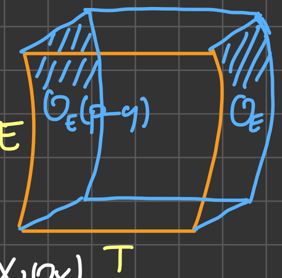
grp $\cong$ $\mathbb{Z}^{2n}$; $\cong$ homeo S^1

$\operatorname{Pic}^0 X$: topologically trivial holo line bundles

$\mathbb{Z}$: const. sheaf in analytic topo

Recall
$$\begin{tikzcd}
\mathbb{C}^n \cong V = T_0X \arrow[d, "\pi"] \\
X
\end{tikzcd}$$
univ. cov.
$\ker = U \cong \mathbb{Z}^{2n}$

Example: E elliptic curve $\mathcal{O}_E(p-q) \xrightarrow[\text{holom.}]{\text{deforms}} \mathcal{O}_E \Rightarrow \mathcal{O}_E(p-q) \cong \mathcal{O}_E$ topologically.



Bottom row: $0 \to \mathbb{Z}_X \to \mathcal{O}_X \longrightarrow \mathcal{O}_X^{\times} \to 0$ sheaves of abelian groups
$f \longmapsto e^{2\pi i f}$

Taking cohomology: $H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to \underset{\substack{\cong \text{ via čech cohom.} \\ \operatorname{Pic}X}}{H^1(X,\mathcal{O}_X^{\times})} \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$

{1 cech cocycle correspond to transition functions of a holo line bundle.

$0 \to \underbrace{\operatorname{Coker}(H^1(\mathbb{Z}_X) \to H^1(\mathcal{O}_X))}_{\text{has a complex structure called dual torus.}} \to \operatorname{Pic}X \xrightarrow{c_1} \ker(H^2(X,\mathbb{Z}) \to H^2(\mathcal{O}_X))$

we will show $H^1(\mathbb{Z}_X) \cong \mathbb{Z}^{2n}$, $H^1(\mathcal{O}_X) \cong \mathbb{C}^n$

holo. line bundles $\xrightarrow[\text{obv. map}]{}$ topological line bundles

Key to the proof: $H^i(X,\mathcal{F})$, $\underbrace{H^i(V,\pi^*\mathcal{F}) = 0}_{\text{for our sheaves}}$ for $i>0 \Rightarrow H^0(V,\pi^*\mathcal{F}) = \Gamma(V,\pi^*\mathcal{F})$

$\Gamma(V,\pi^*\underset{\mathbb{Z}_X,\mathcal{O}_X,\mathcal{O}_X^*}{\mathcal{F}}) \longleftarrow$ U acts on it with isom. of abelian grps $\Rightarrow$ U module $\Rightarrow H^i(X,\mathcal{F}) \overset{\phi^i}{\cong} \underbrace{H^i(U,\Gamma(V,\pi^*\mathcal{F}))}_{\text{group cohomology}}$

Next Goal: Defining ϕ^i & show it is an isomorphism.

Reparation: Group cohomology:

M is a U-module $\Leftrightarrow$ M is $\mathbb{Z}[U]$-module. $H^i(U,M) \overset{\text{def}}{=} \operatorname{Ext}^i_{\mathbb{Z}[U]}(\underset{\text{trivial action of } U}{\mathbb{Z}},M)$ is a definition of grp. cohomology.

Another definition: $H^i(\cdots \to C^j \xrightarrow{\delta^j} C^{j+1} \longrightarrow \ldots)$ and $C^j = \operatorname{Hom}_{\text{set}}(U^{\oplus j},M)$ $\delta^j: C^j \to C^{j+1}$, $f \mapsto \delta f$

(Exercise: the two definitions are the same).

$$(\delta f)(g_0,-,g_j) = g_0.f(g_1,-,g_j) + \sum_{i=0}^{j-1}(-1)^{i+1} f(g_0,-,g_i\cdot g_{i+1},g_{i+2},-,g_j) + (-1)^{j+2} f(g_0,-,g_{j-1})$$

Another definition: Derived functor of
$$\underset{\{U\text{-modules}\}}{M} \longrightarrow \underset{\{\text{ab gps}\}}{M^U = \{m \in M \mid \forall u \in U\ u.m = m\}}.$$

ϕ^j is an isomorphism:

$\underbrace{\pi^*\mathcal{F}}_{\text{sheaf on } V} \to \underbrace{\Gamma(V,\pi^*\mathcal{F})}_{U\text{-module}} \longrightarrow \Gamma(V,\pi^*\mathcal{F})^U = \Gamma(X,\mathcal{F}) \Rightarrow \Gamma(X,-) \cong (-)^U \circ \Gamma(V,\pi^*(-))$ as functors.

$\overset{\Longrightarrow}{\text{(general nonsense)}}$ $R\Gamma(X,-) = R(-)^U \circ R\Gamma(V,\pi^*(-)) \xRightarrow[H^i(V,\pi^*\mathcal{F})=0,\ i>0]{} \phi^j$ isomorphism.

Need: $\Gamma(V,\pi^*(\text{inj}))$ is injective.

ϕ^j explicitly: $\phi^j: H^j(U,\Gamma(V,\pi^*\mathcal{F})) \longrightarrow H^j(X,\mathcal{F})$ $f \in C^j = \operatorname{Hom}_{\text{set}}(U^{\oplus j},\Gamma(V,\pi^*\mathcal{F})) \mapsto (\phi^j f)_{i_0 - i_j}$

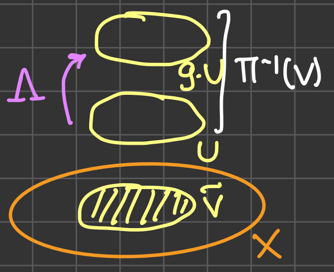
Fix $\underbrace{\text{"nice"}}_{\text{see Lecture 3}}$ cover $\{V_i\}_{i\in I}$ of X

END OF LECTURE 2.

Setup: $\underset{\underset{\mathbb{C}^n}{\shortparallel}}{V} = T_0X \xrightarrow{\pi} \underset{\text{complex torus}}{X}$ surj morph. of Lie groups. $\ker\pi \subseteq V$ discrete subgroup. $\overset{\shortparallel}{\Lambda} \cong \mathbb{Z}^{2n}$

Key to Appel-Humbert: $H^i(\Lambda, \Gamma(\pi^*F)) \xrightarrow[\text{group coho}]{\phi^i} H^i(F)$ we will show that ϕ^i is an isomorphism. The sheaves that we are dealing are $F = \mathbb{Z}_X, \mathcal{O}_X, \mathcal{O}_X^*$ after pulling back $H^i(\pi^*F) = 0$ for $\mathbb{Z}_X, \mathcal{O}_X$, by short exact sequence $H^i(\pi^*\mathcal{O}_X^*) = 0$. We need this for the iso

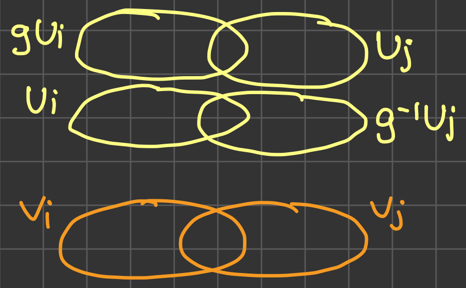
Λ-action on π^*F:



π locally biholo $\Rightarrow \pi^{-1}F|_{g.U} = \pi^*F|_{gU} \overset{\text{canonically}}{\cong} F|_V$ but also $F|_V \cong \pi^*F|_U$, the action of Λ on π^*F is $\pi^*F|_U \longrightarrow \pi^*F|_{g.U}$.

* We want to use Čech here $\{V_i\} \subseteq_{\text{open cover}} X$ st $V_{i_0,\dots,i_j} \overset{\text{def}}{=} V_{i_0} \cap \dots \cap V_{i_j}$ contractible, $\pi^{-1}V_i = \coprod_{g\in\Lambda} gU_i$ st $\pi : gU_i \to V_i$ biholo. $\forall i,j$ $U_i \cap gU_j \neq \emptyset$ $\exists$ at most one $g \in \Lambda$ (Homework : it exists) define $g_{ij} = g$ in that case.

Picture for clarification,



Čech complex: $\check{C}^j(F) = \prod_{i_0 < \dots < i_j} F(V_{i_0,\dots,i_j}) \ni f$

$$\check{\delta}^j(f)_{i_0,\dots,i_{j+1}} = \sum_{s=0}^{j+1} (-1)^s f_{i_0,\dots,\hat{i_s},\dots,i_j}|_{V_{i_0,\dots,i_{j+1}}}$$

Definition of Φ^j: $\Phi^j : C^j(\Lambda, \Gamma(\pi^*F)) \longrightarrow \check{C}^j(F)$ st $\delta\phi = \phi\delta$

let $f : \Lambda^j \to \Gamma(\pi^*F)$ define,

$$(\phi^i f)_{i_0,\dots,i_j} = f(g_{i_0 i_1}, g_{i_1 i_2}, \dots, g_{i_{j-1} i_j})|_{\underset{\underset{V_{i_0,\dots,i_j}}{\downarrow \pi}}{U_{i_0} \cap g_{i_0 i_1} U_{i_1} \cap \dots \cap g_{i_0 i_j} U_{i_j}}}$$

We can assume that $V_{i_0,\dots,i_j} \neq \emptyset \Rightarrow g_{i_a i_b}$ exists $\forall 0 \leq a, b \leq j$. $g_{i_a i_b} \cdot g_{i_b i_c} = g_{i_a i_c}$.

* Why ϕ is a cochain homomorphism: Let $U = U_{i_0} \cap g_{i_0 i_1} U_{i_1} \cap \dots \cap g_{i_0 i_{j+1}} U_{i_{j+1}}$

$$(\check{\delta}^j \phi^j f)_{i_0,\dots,i_{j+1}} = (f(g_{i_1 i_2}, \dots, g_{i_j i_{j+1}}))|_{g_{i_0 i_1}^{-1} U} + \sum_{s=0}^{j-1} (-1)^{s+1} f(g_{i_0 i_1}, \dots, g_{i_s i_{s+2}}, \dots, g_{i_j i_{j+1}})|_U$$

$$+ (-1)^{j+1} f(g_{i_0 i_1}, \dots, g_{i_{j-1}, i_{j+1}})|_U.$$ Notice that the first term can be written as

$$(f(g_{i_1 i_2}, \dots, g_{i_j i_{j+1}}))|_{g_{i_0 i_1}^{-1} U} = g_{i_0 i_1} f(g_{i_1 i_2}, \dots, g_{i_j, i_{j+1}})|_U$$ after rewriting the first term in this way, we see that we get the expression of the boundary map.

Want to show that ϕ^j is an isomorphism. Induction on j: $j = 0 \Rightarrow H^0(\Lambda, \Gamma(\pi^*F)) = \Gamma(\pi^*F)^\Lambda$ and $H^0(F) = \Gamma(F)$ and ϕ^0 is the obvious isomorphism.

$j > 0$: $0 \to F \to \underset{\text{injective}}{F'} \to \underset{\text{coker}}{F''} \to 0.^{(*)}$ We assumed $H^i(\pi^*F) = 0$ for $i > 0$ we have the same for F', F''. F' injective $\Rightarrow F'$ flasque $\Rightarrow \pi^*F'$ is also flasque. as π is a local biholo $\Rightarrow H^i(\pi^*F') = 0$ $\forall i > 0$ by exact sequence, $H^i(\pi^*F'') = 0$ $\forall i > 0$.

Also $0 \to \Gamma(\pi^*F) \to \Gamma(\pi^*F') \to \Gamma(\pi^*F'') \to 0^{(**)}$ is exact. (π^* is exact as π local biholo)

Applying sheaf cohomology to $(*)$ and group cohomology to $(**)$ we get:

$$\begin{tikzcd}
H^{j-1}(\Lambda, \Gamma(\pi^*F')) \arrow[r] \arrow[d, "\phi"] & H^{j-1}(\Lambda, \Gamma(\pi^*F'')) \arrow[r] \arrow[d, "\phi"] & H^j(\Lambda, \Gamma(\pi^*F)) \arrow[r] \arrow[d, "\phi^j"] & H^j(\Lambda, \Gamma(\pi^*F')) \arrow[d, "\phi"] \\
H^{j-1}(F') \arrow[r] & H^{j-1}(F'') \arrow[r] & H^j(F) \arrow[r] & H^j(F')
\end{tikzcd}$$

Homework: This diagram commutes.

Lemma: $\Gamma(\pi^*F')$ is an injective G-module

This Lemma implies that the first and the last terms of the top row are $0 \Rightarrow \phi^j$ is an iso.

Proof: M is a G-module $\Rightarrow M_V$ is a G-constant sheaf $\Rightarrow (\pi_* M_V) \curvearrowleft G$

$\mathrm{Hom}_G(M, \Gamma(\pi^*F')) \cong \mathrm{Hom}_G(M_V, \pi^*F') \cong \mathrm{Hom}((\pi_* M_V)^\Lambda, F')$

as all sections of M_V are global.

Apply this to M_1 & M_2 of G-module. $\mathcal{F}$ injective $\Rightarrow$

$$\begin{tikzcd} \mathrm{Hom}((\pi_* M_2)^{\Lambda}, F') \arrow[r, two heads] \arrow[d, phantom, "\cong"] & \mathrm{Hom}((\pi_* M_1)^{\Lambda}, F') \arrow[d, phantom, "\cong"] \\ \mathrm{Hom}_G(M_2, \Gamma(\pi^* F')) \arrow[r, two heads] & \mathrm{Hom}(M_1, \Gamma(\pi^* F')) \end{tikzcd}$$

$\Rightarrow \Gamma(\pi^* F')$ is injective. $\square$

Line bundles:

$$\begin{tikzcd} L \text{ line bunde on } X \arrow[r, bend left, "\text{local sect.}"] & \mathcal{L} \text{ inv. sheaf.} \arrow[l, bend left, "\text{glwing.}"] \end{tikzcd}$$

$$\begin{tikzcd} \pi^* L & \mathbb{C}\times V \cong \mathcal{O}_V \arrow[l, "\exists \cong"'] \\ \Lambda \text{ acts linearly} \arrow[u] \arrow[r, leftrightarrow] & \Lambda \text{ acts linearly} \arrow[u] \end{tikzcd}$$

$\mathrm{Pic}(V) = 1$.

The action of Λ: $\forall g \in \Lambda$ $\exists \alpha_g : \mathbb{C}\times V \to \mathbb{C}\times V$ compatible with Λ action on V.

i.e

$$\begin{tikzcd} \mathbb{C}\times V \arrow[r, "\alpha_g"] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}\times V \arrow[d] \\ V \arrow[r, "g"'] & V \end{tikzcd}$$

& α_g linear in the first coordinate,

$\alpha_g(-,z) : \mathbb{C}\times\{z\} \longrightarrow \mathbb{C}\times\{g+z\}$

$v \longmapsto e_g(z).g$ with $e_g(z) \in \mathbb{C}^\times$

e_g is a holo function. $\Rightarrow e_g \in \Gamma(\mathcal{O}_V^\times) = H^\times$] Λ-module.

$\Rightarrow (e : \Lambda \to H^\times) \in H^1(\Lambda, H^\times)$ as soon as it is a cocycle. Need: $\delta e = 0$.

$(\delta e)_{(g,g')} = (g.e_{g'}).(e_{gg'})^{-1}.e_g \overset{?}{=} 0$. Note that $\alpha_{gg'} = \alpha_g \circ \alpha_{g'}$,

$\alpha_{gg'}(t,z) = (e_{gg'}(z).t, z+gg')$, $\alpha_g(\alpha_{g'}(t,z)) = \alpha_g(e_{g'}(z).t, z+g') = e_g(z+g')(e_{g'}(z).t, z+gg')$

after reversing the roles of g & g' we obtain the desired inequality.

So we get
$$\begin{tikzcd} H^1(X, \mathcal{O}_X^\times) \arrow[r, bend left] & H^1(\Lambda, H^\times) \arrow[l, bend left, "\phi^1"] \end{tikzcd}$$
indeed the inverse of Φ^1 is what we just defined

END OF LECTURE 3.

Goal: Appel – Humbert:

Recall the exact sequence:

$0 \to \mathrm{Pic}^0 X \to \mathrm{Pic} X \to \ker(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X))$ in fact $H^2(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \underbrace{H^{2,0}}_{\text{as } \mathbb{C} \text{ vs.}} \oplus H^{1,1} \oplus H^{0,2}$

$\xleftarrow{\quad} H^{0,2}$ (Hodge theory). conjugate

Let $H = \Gamma(\pi^* \mathcal{O}_X) = \Gamma(\mathcal{O}_V)$, $H^\times = \Gamma(\pi^* \mathcal{O}_X^\times) = \Gamma(\mathcal{O}_V^\times)$

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0 X \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Pic} X \arrow[r] \arrow[d, phantom, "\cong"] & \ker(H^2(X,\mathbb{Z}) \to H^2(X, \mathcal{O}_X)) \\ 0 \arrow[r] & \mathrm{coker}(H^1(X,\mathbb{Z}) \to H^1(X, \mathcal{O}_X)) \arrow[r] & H^1(X, \mathcal{O}_X^\times) & \\ 0 \arrow[r] & \mathrm{coker}(H^1(\Lambda,\mathbb{Z}) \to H^1(\Lambda, H)) \arrow[r] \arrow[u, "\varphi", "\cong"'] & H^1(\Lambda, H^*) \arrow[r, "\delta"] \arrow[u, "\varphi", "\cong"'] \arrow[u, bend left, leftarrow, "\varphi^{-1}"] & \ker(H^2(\Lambda,\mathbb{Z}) \to H^2(\Lambda, H^*)) \arrow[r] \arrow[uu, "\varphi", "\cong"'] & 0 \quad (*) \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Lambda, H^*) \ni L = e \arrow[r, mapsto] & ? \text{ explicit description} \end{tikzcd}$$

$\varphi^{-1}(L) : \pi^{-1} L \cong \mathbb{C}\times V$, $\Lambda \ni g \mapsto \alpha_g \in \mathrm{Aut}_g(\mathbb{C}\times V)$ $\alpha_g(t,z) \mapsto (\overset{\in \mathbb{C}^\times}{e_g(z)}.t, z+g)$

$e_g \in H^\times \longrightarrow e : \mathrm{Hom}_{set}(\Lambda, H^*) \to \varphi^{-1}(L) = \{e\} \in H^1(\Lambda, H^*)$.

Understanding: $H^*(\Lambda, \mathbb{Z}) \xrightarrow{\varphi} H^*(X, \mathbb{Z})$, let $[h] \in H^1(\Lambda, \mathbb{Z}) \Rightarrow h \in \mathrm{Hom}_{set}(\Lambda, \mathbb{Z})$

$(\varphi h)_{g,\tilde{g}} = \underbrace{\cancel{g.}}_{\text{action is trivial on } \mathbb{Z}} h(\tilde{g}) - h(\tilde{g}) - h(g+\tilde{g}) + h(g) = 0$ thus h is a co-chain $\Rightarrow h \in \mathrm{Hom}_{gp}(\Lambda, \mathbb{Z})$.

$h_g = (\varphi \tilde{h})_g = g\tilde{h} - \tilde{h} = \tilde{h} - \tilde{h} = 0 \Rightarrow$ co-boundary $\Rightarrow 0$.

$\Rightarrow \exists$ canonical $H^1(\Lambda, \mathbb{Z}) = \mathrm{Hom}_{gp}(\Lambda, \mathbb{Z})$.

X is a torus $\Rightarrow X \cong_{diffeo} (S^1)^{2n}$ and $H^*(S^1, \mathbb{Z}) = \Lambda^* H^1(S^1, \mathbb{Z}) \xrightarrow[\text{Comp. w/ Cupprod.}]{\text{Künneth formula}}$

$H^*(X, \mathbb{Z}) = \bigotimes^{2n} \Lambda^* H^1(S^1, \mathbb{Z}) = \Lambda^* H^1(X, \mathbb{Z})$.

$\llcorner$ Künneth $H^1(X, \mathbb{Z}) \cong \bigoplus^{2n} H^1(X, S^1)$

$$\begin{tikzcd} H^*(\Lambda, \mathbb{Z}) \arrow[r, "\varphi \cong"] & H^*(X, \mathbb{Z}) \arrow[d, phantom, "\cong"] \\ \Lambda^* H^1(\Lambda, \mathbb{Z}) \arrow[r, "\Lambda^*(\varphi) \cong"] \arrow[u, "\text{cup product}", "\cong"'] \arrow[d, phantom, "\cong"] & \Lambda^* H^1(X, \mathbb{Z}) \\ \Lambda^* \mathrm{Hom}_{gp}(\Lambda, \mathbb{Z}) & \end{tikzcd}$$

Specialize to $n=2$: $H^2(\Lambda, \mathbb{Z}) \cong \underbrace{\Lambda^2 \underbrace{\mathrm{Hom}_{gp}(\overset{\mathbb{Z}^{2n} \cong}{\Lambda}, \mathbb{Z})}_{\cong_{\text{non canonically}} \mathbb{Z}^{2n}}}_{\{\text{alternating bilinear forms on } \Lambda\}}$

$H^2(\Lambda,\mathbb{Z}) \cong \{\text{alternating bilinear maps } \Lambda \to \mathbb{Z}\}$

$\overset{\cup}{F} \longmapsto \left((g,\tilde{g}) \mapsto F(g,\tilde{g}) - F(\tilde{g},g)\right) = AF$

We go back to (*) this exact sequence comes from the exponential sequence, for $e_U \in H^\times$ can choose f_U such that $e^{2\pi i f_U} = e_U$ the exponential sequence is:

$$0 \to \mathbb{Z} \to H \xrightarrow{f \mapsto e^{2\pi i f}} H^* \to 0 \quad \Rightarrow e \overset{\delta}{\longmapsto} \underbrace{f_{U_2}(z+U_1) - f_{U_1+U_2}(z) + f_{U_1}(z)}_{\text{constant in } z} \in \mathbb{Z}$$

constant in z ⇒ denote by $F(U_1,U_2)$

⇒ we obtain the alternating form $AF(U_1,U_2) = F(U_1,U_2) - F(U_2,U_1) = f_{U_2}(z+U_1) + f_{U_1}(z) - f_{U_1}(z+U_2) - f_{U_2}(z)$.

Corollary: The $\mathbb{R}$-linear extension of an alternating bilinear form $\Lambda \to \mathbb{Z}$ is unique given by $E: V \to \mathbb{R}$. ⇒ $E(x,y) = E(ix,iy)$. Some algebraic lemma gives:

$\{\text{anti-sym. bilinear forms on } V \text{ satisfying } E(x,y) = E(ix,iy)\} \Leftrightarrow \{\text{Hermitian forms}\}$

$\mathrm{im}H \longleftarrow H$

$E \longmapsto E(ix,y) + iE(x,y).$

⇒ AF extends to some E ⇒ $\exists!\, H: V \times V \to \mathbb{C}$ Hermitian st $\mathrm{Im}H = E$

⇒ There is a map

$\ker(H^2(\Lambda,\mathbb{Z}) \to H^2(\Lambda, H^*)) \longrightarrow \{H \text{ Hermitian form on } V \text{ satisfying } H(\Lambda \times \Lambda) \subseteq \mathbb{Z}\}.$

Want to understand how to go back. Fix $H: V \times V \to \mathbb{C}$ Hermitian, want to find f_U corresponding to H. We know $\mathrm{Im}H = f_{U_2}(z+U_1) + f_{U_1}(z) - f_{U_1}(z+U_2) - f_{U_2}(z)$

Note that $f_U(z) = \frac{1}{2i} H(z,U) + \underset{\mathbb{C}}{\overset{\cap}{B_U}}$ works.

Because: $\frac{1}{2i}\big(\underbrace{H(z+U_1,U_2)}_{H(z,U_2)+H(U_1,U_2)} + H(z,U_1) - \underbrace{H(z+U_2,U_1)}_{H(z,U_1)+H(U_2,U_1)} - H(z,U_2)\big) = \frac{1}{2i}\left(H(U_1,U_2) - H(U_2,U_1)\right) = \underset{\in\mathbb{Z}}{\mathrm{Im}H(U_1,U_2)}.$

one more condition: $f_{U_2}(z+U_1) - f_{U_1+U_2}(z) + f_{U_1}(z) \in \mathbb{Z}$ this condition implies $\frac{1}{2}H(U_1,U_2) + i(B_{U_1} + B_{U_2} - B_{U_1+U_2}) \in i\mathbb{Z}$. Introduce $\gamma_U = iB_U - \frac{1}{4}H(U,U)$.

⇒ $\gamma_{U_1} + \gamma_{U_2} - \gamma_{U_1+U_2} + \frac{1}{2}iE(U_1,U_2) \in i\mathbb{Z}$. ⇒ $\mathrm{Re}\gamma$ is a group homomorphism.

$$\begin{tikzcd} \Lambda \arrow[r, "\mathrm{Re}\gamma"] \arrow[d] & (\mathbb{R},+) \\ V \arrow[ur, "\exists!\, \mathbb{C}\text{-lin.}"'] & \end{tikzcd}$$

⇒ $\exists!\, L \in \mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$ st $\mathrm{Re}L = \mathrm{Re}\gamma$. Can replace γ by $\gamma - L$.

⇒ we may assume that $\mathrm{Re}\gamma = 0$.

Define $\alpha(U) := e^{2\pi\gamma}$ ⇒ $|\alpha(U)| = 1$. $\frac{\alpha(U_1+U_2)}{\alpha(U_1)\cdot\alpha(U_2)} = e^{\pi i E(U_1,U_2)}$ but $E(U_1,U_2) \in \mathbb{Z}$ ⇒ $e^{\pi i E(U_1,U_2)} \in \{\pm 1\}$.

Note that there is always a solution.

$$e_U(z) = e^{2\pi i f_U(z)} = e^{2\pi i\left(\frac{1}{2\pi i} H(z,U) + B_U\right)} = e^{\pi i H(z,U)} \overset{\alpha(U)}{=} e^{\pi H(z,U) + 2\pi\frac{1}{4}H(U,U)}$$

HW: Is there a Hermitian metric on $\mathbb{C} \times V$ invariant under the Λ-action such that the curvature is H?

HW: α always exists. HW: This is a cocycle

This shows that the map $H^1(\Lambda, H^\times) \twoheadrightarrow \{H: \text{Hermitian form on } V \text{ st } H(\Lambda\times\Lambda) \subseteq \mathbb{Z}\}$ is surjective an the kernel is the same with the map, $H^1(\Lambda, H^\times) \twoheadrightarrow \ker(H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X))$.

⇒ these are isomorphic.

$$\begin{tikzcd}
\Rightarrow\ 0 \arrow[r] & \left\{\alpha \,\middle|\, \begin{array}{l}\alpha:\Lambda\to\{z\in\mathbb{C}:|z|=1\}\\ \text{group homo}\end{array}\right\} \arrow[r] \arrow[d] & \left\{(\alpha,H) \,\middle|\, \begin{array}{l}\bullet\ H \text{ hermitian}\\ \bullet\ H(\Lambda\times\Lambda)\subseteq\mathbb{Z}\\ \bullet\ \alpha:\Lambda\to\{z\in\mathbb{C}:|z|=1\}\end{array}\right\} \arrow[r] \arrow[d] & \{H \text{ as before}\} \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{coker}(H^1(\Lambda,\mathbb{Z})\to H^1(\Lambda,H)) \arrow[r] & H^1(\Lambda,H^*) \arrow[r, "\delta"] & \ker(H^2(\Lambda,\mathbb{Z})\to H^2(\Lambda,H^*)) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

END OF LECTURE 4

Where are we:

Thm: (extract of Appel Humbert): X complex torus, $V=T_0X \xrightarrow{\pi} X$ $\ker\pi=\Lambda$.

$\operatorname{Pic}X \Leftrightarrow \{(\alpha,H) \mid H:V\times V\to\mathbb{C}$ hermitian, $\operatorname{Im}H(\Lambda\times\Lambda)\subseteq\mathbb{Z}$ and $\alpha:\Lambda\to\mathbb{C}_1^\times=\{z\in\mathbb{C} \mid |z|=1\}$
$\alpha(g+g')/\alpha(g)\alpha(g')=e^{\pi i\operatorname{Im}H(g,g')}\}$

The correspondence is given by, $(\alpha,H)\longmapsto e_u\in H^1(\Lambda,\Gamma(\pi^*\mathcal{O}_X^\times))$ such that
$e_u(z)=\alpha(u).e^{\pi H(z,u)+\frac{1}{2}\pi H(u,u)}$

and from e_u we obtain the line bundle: $L(H,\alpha)=\mathbb{C}\times V/_{(t,z)=(e_u(z).t,\,z+u)}$ on $V/\Lambda=X$.

Filling gaps from last time:

*** Why α exists?**

$\operatorname{Im}H:\Lambda\times\Lambda\to\mathbb{Z}$ and $\Lambda\cong\mathbb{Z}^{2n}$ $\overset{\text{choose a basis}}{\Longrightarrow}$ $(a_{ij})\in\operatorname{Mat}(\mathbb{Z},2n\times2n)\Rightarrow(|a_{ij}|)$ gives a map $S:\Lambda\times\Lambda\to\mathbb{Z}$ bilinear symmetric. For $g,g'\in\Lambda$ $S(g,g')-\operatorname{Im}H(g,g')\in2\mathbb{Z}$ then we can define

$$\alpha(g)=e^{1/2\,i\pi S(g,g)}$$

Verification: $\dfrac{\alpha(g_1+g_2)}{\alpha(g_1).\alpha(g_2)}=e^{i\pi S(g_1,g_2)}=e^{i\pi\operatorname{Im}H(g_1,g_2)}$

*** Compatibility with complex structure: $E(x,y)=E(ix,iy)$:**

Let

$$\begin{tikzcd}
\mathcal{L}\in\operatorname{Pic}(X)\cong H^1(X,\mathcal{O}_X^\times) \arrow[r, "\delta"] \arrow[d, "\cong\,\Phi"'] & \ker(H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O}_X)) \arrow[d, "\cong\,\Phi"'] \\
H^1(\Lambda,\Gamma(\pi^*\mathcal{O}_X^\times)) \arrow[r, "\delta"] & \ker(H^2(\Lambda,\mathbb{Z})\to \_\_\_) \arrow[d, phantom, "\cong"] \\
 & \Lambda^2H^1(\Lambda,\mathbb{Z})\cong\Lambda^2\operatorname{Hom}_{gp}(\Lambda,\mathbb{Z})\ni E
\end{tikzcd}$$

So from $\mathcal{L}$ we obtain E and if E satisfies $E(x,y)=E(ix,iy)$ then we obtain a Hermitian form $H(x,y)=E(ix,y)+iE(x,y)$.

Proof: (basic idea) $E\in H^2(X,\mathbb{Z})$ & use E is real (stable under conjugation) and that it is in the $\ker(H^2(X,\mathbb{Z})\to H^2(X,\mathcal{O}_X))$. (Hodge theory).

We can embed,

$$\begin{tikzcd}
\operatorname{Hom}_{gp}(\Lambda,\mathbb{Z}) \arrow[r, hook] \arrow[ddd, "\cong"] & \operatorname{Hom}_{\mathbb{R}}(V,\mathbb{C})\cong T\oplus\overline{T} \arrow[rr, "pr_{\overline{T}}"] \arrow[dd, "\cong"'] \arrow[d, phantom, "\ni"] & & \overline{T} \arrow[r, phantom, "\ni"] \arrow[dd, "\cong"'] & \psi \arrow[dd, mapsto] \\
 & \Phi \arrow[d, mapsto] & & & \\
 & h^1(\mathcal{E}^\bullet) \quad d\Phi \arrow[d, phantom, "\cong"] & & h^1(\mathcal{E}^{0,1}) \arrow[d, phantom, "\cong"] & d\psi \\
H^1(X,\mathbb{Z}) \arrow[r] & H^1(X,\mathbb{C}) \arrow[rr] & & H^1(X,\mathcal{O}_X) &
\end{tikzcd}$$

page 3-12, Mumford

where $T=\operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$ and $\overline{T}=\operatorname{Hom}_{\mathbb{C}\text{-anti}}(V,\mathbb{C})$ and $\mathcal{E}^\bullet$ is the complex of complex valued C^∞-diff forms.

The isomorphisms on the above diagram are compatible with cup product

$$\begin{tikzcd}
E \arrow[d, phantom, sloped, "\in"] \arrow[r, mapsto] & E \text{ st } E=\overline{E} \arrow[r, mapsto] & 0 \arrow[dl, bend right, "\text{both are zero for } E"'] \\
\Lambda^2\mathrm{Hom}_{gp}(\Lambda,\mathbb{Z}) \arrow[r, hook] \arrow[dd, "\cong"'] & \Lambda^2(T\oplus\overline{T}) \arrow[r] \arrow[d, phantom, sloped, "\cong"] & \overline{T} \arrow[dd, "\cong"] \\
 & \Lambda^2T\oplus\Lambda^2\overline{T}\oplus(T\otimes\overline{T}) \arrow[d] & \\
H^2(X,\mathbb{Z}) \arrow[r] & H^2(X,\mathbb{C}) \arrow[r] & H^2(X,\mathcal{O}_X)
\end{tikzcd}$$

$\Rightarrow E=\sum_j \overset{\overset{T}{\rotatebox{90}{$\in$}}}{\varphi_j}\cdot\overset{\overset{\overline{T}}{\rotatebox{90}{$\in$}}}{\psi_j}$

$\varphi_j\cdot\psi_j(ix,iy))=\varphi_j(ix)\cdot\psi_j(iy)$

$=i\cdot(-i)\varphi_j(x)\cdot\psi_j(y)=\varphi_j(x)\cdot\psi_j(x).$

$\square.$

Question: How can it be that X has no topologically non trivial line bundles ($\Rightarrow X$ is not algebraic as an alg. projective)

Obstruction: $\Lambda^2\mathrm{Hom}_{gp}(\Lambda,\mathbb{Z})\hookrightarrow\Lambda^2\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})=\Lambda^2T$

$[\mathrm{Hom}_{gp}(\Lambda,\mathbb{Z})\cong\mathbb{Z}^{2n}\underset{\text{lattice}}{\hookrightarrow}\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C})=T]$

Because then the Λ^2T coordinate is non zero $\forall$ element of $H^2(X,\mathbb{Z})$.

Lemma: $\Lambda\subseteq V\cong\mathbb{C}^n$ where Λ is general $\Rightarrow\Lambda^n\Lambda\xhookrightarrow{\phi}\Lambda^nV=\mathbb{C}$

Proof: $\omega_1,-,\omega_{2n}\in\mathbb{C}^n$ a basis of Λ in a basis of V.

$\phi(e_{i_1}\wedge\_\wedge e_{i_n})=\det[\omega_{i_1},-,\omega_{i_n}]$

Need:

$$\sum_{\substack{I\subseteq\{1,-,2n\}\\|I|=n}}\overset{\overset{\mathbb{Z}}{\rotatebox{90}{$\in$}}}{a_I}\cdot\wedge e_I\overset{\phi}{\longmapsto}\sum a_I\det(\omega_I)$$

but we can choose $\omega_1,-,\omega_{2n}$ alg. independent $\Rightarrow$ this is not zero $\square$

Example: $a,b,c,d\in\mathbb{C}$ alg. indep. $/\mathbb{Q}$ $\Lambda=\mathbb{Z}\begin{bmatrix}1\\0\end{bmatrix}\oplus\mathbb{Z}\begin{bmatrix}0\\1\end{bmatrix}\oplus\mathbb{Z}\begin{bmatrix}a\\b\end{bmatrix}\oplus\mathbb{Z}\begin{bmatrix}c\\d\end{bmatrix}\Rightarrow V/\Lambda$ not alg.
Need $\Lambda^2\Lambda\hookrightarrow\Lambda^2V$ which is true when we workout the wedge products.

Final Question:

Theorem (Lefschetz): $L(H,\alpha)$ $\underline{\text{ample}}\Leftrightarrow H$ is positive definite.
↳ Some tensor power induces an embedding

$s_0,-,s_r\in H^0(X,L)$ globally generating $\Rightarrow X\ni x\longmapsto[s_0(x):\_:s_r(x)]\in\mathbb{P}^r$

Q: $H^0(X,L(H,\alpha))=?$

Let $s\in H^0(X,L(H,\alpha))\longleftrightarrow\Theta:V\to\mathbb{C}$ st $\Theta(z+g)=e_g(z)\cdot\Theta(z)=\alpha(g)\cdot e^{\pi H(z,g)+\frac{1}{2}\pi H(g,g)}\Theta(z)$ for all $z\in V$ $g\in\Lambda$.

Theorem: H positive definite $\Rightarrow\dim_{\mathbb{C}}H^0(X,L(H,\alpha))=\sqrt{|\det E|}$ where $E=\mathrm{Im}H:\Lambda\times\Lambda\to\mathbb{Z}$

Proof: Write down a basis by hand by modifying Θ such that it becomes periodic for some $\Lambda'\subseteq\Lambda$

Definition of the sublattice: There is $\Lambda'\subseteq\Lambda$ st

- $\Lambda'\cong\mathbb{Z}^n$
- $E(\Lambda'\times\Lambda')=0$
- $W=\mathbb{R}.\Lambda'\Rightarrow W\cap\Lambda=\Lambda'$

(Recall: $H(x,y)=E(ix,y)+iE(x,y)$) $\Rightarrow H|_{W\cap iW}=0$ but H is positive definite $\Rightarrow W\cap iW=\{0\}$.

$\Rightarrow V=W\oplus iW$ as $\mathbb{R}$-vector spaces.

$\Rightarrow$ for $H|_W$ real symmetric $\exists!$ extension B to a complex symmetric bilinear form.

$\Rightarrow\underbrace{H-B}_{\text{bilinear on }V}|_W=0$ Consider the $\Theta^*(z)=e^{-1/2\pi B(z,z)}\cdot\Theta(z)\underset{\text{for }g\in\Lambda'}{\Rightarrow}\Theta^*(z+g')=\underbrace{\alpha(g')}_{\text{actual hom}\Rightarrow e^{2\pi i\lambda(-)}}\cdot\Theta^*(z)$

$\Rightarrow e^{-2\pi i\lambda(-)}\cdot\Theta^*(z)$ is periodic.